Nonexistence of isolated charged monopoles in unified field theories

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Abstract. The possibility of the existence of isolated electric charge giving rise to a spherically symmetric electrostatic field has been considered in each of the unified field theories of Einstein, Schrödinger and Bonnor. It has been proved that none of these theories admits such a possibility.

1. Introduction

On the assumption that the source of gravitational and electromagnetic effects produces a spherically symmetric field, Papapetrou (1948) has shown that the field tensor

$$g_{ij} = \begin{pmatrix} -\alpha & 0 & 0 & \omega \\ 0 & -\beta & f\sin\theta & 0 \\ 0 & -f\sin\theta & -\beta\sin^2\theta & 0 \\ -\omega & 0 & 0 & \gamma \end{pmatrix} \tag{1.1}$$

represents the most general total field in spherical polar coordinates $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$, $x^4 = t$. Equation (1.1) has been of considerable importance in connection with the investigations of static spherically symmetric solutions in some unified field theories. Papapetrou (1948) examined (1.1) in the unified field theory of Schrödinger (1947) and gave exact solutions for two cases, namely (i) the purely electric case $f \neq 0$, $\omega = 0$ and (ii) the purely magnetic case $f = 0$, $\omega \neq 0$ which in the absence of the electromagnetic field and for $\lambda = 0$ reduce to Schwarzschild's solution in general relativity. Wyman (1950) and Bonnor (1951) obtained respectively real and complex solutions for the above two cases in the unified field theory of Einstein and Straus (1946) and discussed their physical significance. In order to avoid the physical drawback of the above solutions representing an isolated magnetic pole corresponding to case (ii) Wyman (1950) and Tonnelat (1955) felt that a more satisfactory conclusion could be drawn in the general case $f \neq 0$, $\omega \neq 0$. Accordingly Takeno *et al* (1951), Bonnor (1952) and Bandyopadhyay (1960) obtained general solutions for this case in the field theories

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of Schrödinger (1947), Einstein (1951) and Einstein and Straus (1946) respectively. Ghosh (1955, 1956) obtained a more general solution in the theory of Einstein (1951) of which Bonnor's solution (1952) happens to be a special case. Later on Godart (1961) and Vanstone (1962) gave general solutions to the field equations of Schrödinger (1947) and Einstein and Straus (1946) which contain the solutions of Papapetrou (1948), Wyman (1950) and Bonnor (1951) as special cases. None of these attempts have yielded physically significant results as the solutions obtained are not singularity free and they do not rule out the existence of a magnetic pole. Bonnor (1952) gave up all hopes for a physical solution in the general case and remarked that the only static spherically symmetric solutions likely to have any physical significance are those corresponding to an electric field alone. Ikeda (1954) proposed new boundary conditions to the solutions in the case of an electrostatic field by introducing electrostatic potential in the set up of unified field theories. Ikeda (1955) further showed that there exists no regular static spherically symmetric solution in the magnetic case satisfying his new boundary conditions which in turn proved the nonexistence of a single magnetic pole in the unified field theory of Einstein (1953). This result gave some indications to the physical base to a unified field theory. The question whether there exists a nontrivial static spherically symmetric solution that may represent the external field of an isolated charged monopole in the unified field theories is still to be answered. The exact solutions of Einstein's and Schrödinger's unified field theories given by Papapetrou (1948), Wyman (1950) and Bonnor (1951) in the case of the electrostatic field correspond to continuous charged distributions though the fields tend asymptotically to that of a point charge in classical theory. In the case of Bonnor's theory (1954) the corresponding solution of the linear approximation to the field equations gives a charge density which decreases exponentially with distance (Abrol 1957). It was Moffat (1957b) who first obtained the spherically symmetric solution representing the field of an isolated charged particle at rest in his unified field theory (Moffat 1957a). One expects to get a Moffat-like solution in the other theories also. Our present investigation is to explore the possibility of the existence of isolated charged poles in the nonsymmetric field theories of Einstein (1953), Schrödinger (1947) and Bonnor (1954).

The field equations considered here are as follows:

$$g_{ij,k}-g_{sj}\Gamma^s_{ik}-g_{is}\Gamma^s_{kj} = 0 \tag{1.2}$$

$$\Gamma^s_{\underset{\vee}{is}} = 0 \tag{1.3}$$

$$R_{\underline{ij}}-\lambda g_{\underline{ij}}+p^2 U_{\underline{ij}} = 0 \tag{1.4}$$

$$R_{[\underset{\vee}{ij},k]}-\lambda g_{[\underset{\vee}{ij},k]}+p^2 U_{[\underset{\vee}{ij},k]} = 0. \tag{1.5}$$

Here p and λ are constants. R_{ij} is the usual contracted curvature tensor given by

$$R_{ij} = -\Gamma^s_{ij,s}+\Gamma^s_{is,j}+\Gamma^s_{iu}\Gamma^u_{sj}-\Gamma^s_{us}\Gamma^u_{ij} \tag{1.6}$$

and

$$U_{ij} = g_{\underset{\vee}{ji}}-g^{\underset{\vee}{mn}}g_{im}g_{nj}+\tfrac{1}{2}g^{\underset{\vee}{mn}}g_{nm}g_{ij}. \tag{1.7}$$

By taking $p = \lambda = 0$ in (1.2) to (1.5) we get Einstein's field equations. We get Schrödinger's and Bonnor's field equations by taking $p = 0$ and $\lambda = 0$ respectively.

2. Preliminary calculations

The total field considered here is given by

$$g_{ij} = \begin{pmatrix} -\alpha & 0 & 0 & 0 \\ 0 & -\beta & f\sin\theta & 0 \\ 0 & -f\sin\theta & -\beta\sin^2\theta & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix} \tag{2.1}$$

where α, β, γ and f are functions of r alone. The contravariant g^{ij} is defined by (Einstein 1953)

$$g_{ik}g^{jk} = \delta^j_i.$$

The nonvanishing components of g^{ij} are

$$g^{11} = -\frac{1}{\alpha} \qquad g^{22} = \sin^2\theta\ g^{33} = -\frac{\beta}{\beta^2+f^2}$$
$$g^{44} = \frac{1}{\gamma} \qquad g^{\underset{\vee}{23}} = \frac{f}{(\beta^2+f^2)\sin\theta}. \tag{2.2}$$

The charge–current vector density is defined by (Einstein 1953)

$$j^s = \tfrac{1}{6}\eta^{ikls}I_{ikl} \tag{2.3}$$

where η^{ikls} is Levi-Civita's tensor density antisymmetric in all indices and

$$I_{ikl} = g_{\underset{\vee}{ik},l} + g_{\underset{\vee}{kl},i} + g_{\underset{\vee}{li},k}.$$

From (2.1) and (2.3) we find that the charge density is

$$j^4 = f'\sin\theta \tag{2.4}$$

where a prime denotes differentiation with respect to r. The external field of the isolated charged monopole is the vacuum electrostatic field characterized by

$$j^4 = 0. \tag{2.5}$$

Hence it follows from (2.4) that

$$f = \text{constant}. \tag{2.6}$$

The nonvanishing components of R_{ij} as calculated from (1.6) using (2.1), (2.6) and (1.2) are given by

$$R_{11} = A' + \frac{1}{2}\left(\frac{\gamma'}{\gamma}\right)' + \frac{1}{2}(A^2+B^2) + \frac{\gamma'}{4\gamma}\left(\frac{\gamma'}{\gamma} - \frac{\alpha'}{\alpha}\right) - \frac{\alpha'}{2\alpha}A$$

$$R_{22} = R_{33}\operatorname{cosec}^2\theta = \left(\frac{\beta A}{2\alpha}\right)' - f\left(\frac{B}{2\alpha}\right)' - \frac{B}{2\alpha}(fA+\beta B) + \frac{\beta A - fB}{2\alpha}\left(\frac{\alpha'}{2\alpha} + \frac{\gamma'}{2\gamma}\right) - 1$$

$$R_{44} = -\frac{1}{2}\left(\frac{\gamma'}{\alpha}\right)' + \frac{\gamma'}{2\alpha}\left(\frac{\gamma'}{2\gamma} - \frac{\alpha'}{2\alpha} - A\right)$$

$$R_{\underset{\smile}{23}} = \sin\theta\left\{-\left(\frac{\beta B}{2\alpha}\right)' - f\left(\frac{A}{2\alpha}\right)' + \frac{B}{2\alpha}(fB-\beta A) - \frac{1}{2\alpha}(fA+\beta B)\left(\frac{\alpha'}{2\alpha}+\frac{\gamma'}{2\gamma}\right)\right\} \tag{2.7}$$

where

$$A = \frac{\beta\beta'}{\beta^2+f^2} \qquad B = \frac{f\beta'}{\beta^2+f^2}. \tag{2.8}$$

3. Field equations in Einstein's theory

By taking $p = \lambda = 0$ in (1.4) and (1.5) we get Einstein's field equations in the form:

$$R_{11} = 0 \qquad R_{22} = 0 \qquad R_{44} = 0 \qquad R_{\underset{\smile}{23}} = c\sin\theta \tag{3.1}$$

where c is an arbitrary constant. Taking proper linear combinations (Papapetrou 1948) of these we find that the simplified field equations are

$$A' + \frac{1}{2}(A^2+B^2) - \frac{1}{2}A\left(\frac{\alpha'}{\alpha}+\frac{\gamma'}{\gamma}\right) = 0 \tag{3.2a}$$

$$\frac{\gamma''}{\gamma} - \frac{1}{2}\frac{\gamma'}{\gamma}\left(\frac{\alpha'}{\alpha}+\frac{\gamma'}{\gamma}\right) + A + \frac{\gamma'}{\gamma} = 0 \tag{3.2b}$$

$$\beta'' - \frac{1}{2}\beta'\left(\frac{\alpha'}{\alpha}-\frac{\gamma'}{\gamma}\right) + \frac{2\alpha}{\beta^2+f^2}(2\beta fc - \beta^2 + f^2) = 0 \tag{3.2c}$$

$$-\beta B - \frac{2\alpha}{\beta^2+f^2}(c\beta^2 - cf^2 + 2\beta f) = 0. \tag{3.2d}$$

Equations (3.2*a*) and (3.2*b*) can be integrated at once giving

$$A^2 = \frac{c_1\alpha\gamma}{\beta} \tag{3.3}$$

and

$$(\gamma')^2 = \frac{c_2\alpha\gamma}{\beta^2+f^2} \tag{3.4}$$

c_1 and c_2 being arbitrary constants. From (2.8), (3.3) and (3.4) we obtain

$$\gamma' = \left(\frac{c_2}{c_1}\right)^{1/2}\left(\frac{\beta}{\beta^2+f^2}\right)^{3/2}\beta'. \tag{3.5}$$

Also from (3.2*d*) and (2.8) we have

$$\alpha = -\frac{f(\beta')^2}{2(c\beta^2 - cf^2 + 2\beta f)}. \tag{3.6}$$

From (3.3) and (3.6) we find that

$$\gamma = -\frac{2\beta^3(c\beta^2 - cf^2 + 2\beta f)}{c_1 f(\beta^2+f^2)^2}. \tag{3.7}$$

Differentiating (3.7) and comparing the result with (3.5) one gets

$$\beta'\left\{\frac{2\beta^2}{c_1 f(\beta^2+f^2)^3}(c\beta^4+6cf^2\beta^2+8f^3\beta-3cf^4)+\left(\frac{c_2}{c_1}\right)^{1/2}\left(\frac{\beta}{\beta^2+f^2}\right)^{3/2}\right\} = 0 \tag{3.8}$$

which gives either

$$\beta' = 0 \tag{3.9}$$

or

$$\frac{\beta^{1/2}}{(\beta^2+f^2)^{3/2}}(c\beta^4+6cf^2\beta^2+8f^3\beta-3cf^4) = -\frac{1}{2}f(c_1c_2)^{1/2}. \tag{3.10}$$

We find in either case that β = constant. This leads to $\alpha = 0$ which is physically inadmissible.

4. Field equations in Schrödinger's theory

In addition to equation (3.2*a*) the other field equations in Schrödinger's theory are

$$\frac{\gamma''}{\gamma}-\frac{\gamma'}{2\gamma}\left(\frac{\alpha'}{\alpha}+\frac{\gamma'}{\gamma}\right)+A\frac{\gamma'}{\gamma}+2\lambda\alpha = 0 \tag{4.1a}$$

$$\beta''-\frac{1}{2}\beta'\left(\frac{\alpha'}{\alpha}-\frac{\gamma'}{\gamma}\right)+\frac{2\alpha}{\beta^2+f^2}(\lambda\beta^2-\beta^2-\lambda\beta f+2\beta fc+f^2) = 0 \tag{4.1b}$$

$$-f(\beta')^2+2\alpha(2\lambda\beta^2 f-2\beta f-c\beta^2+cf^2) = 0. \tag{4.1c}$$

From (4.1*c*) we find that

$$\alpha = \frac{f(\beta')^2}{2(2\lambda\beta^2 f-2\beta f-c\beta^2+cf^2)}. \tag{4.2}$$

Also from (3.3) and (4.2) we have

$$\gamma = \frac{2\beta^3(2\lambda\beta^2 f-2\beta f-c\beta^2+cf^2)}{c_1 f(\beta^2+f^2)^2}. \tag{4.3}$$

Substituting for α and γ in (4.1*a*) we find on simplification

$$\frac{\beta(\beta')^2}{c_1 f(\beta^2+f^2)}\{(8\lambda f-3c)\beta^6-3f^2\beta^4(4\lambda f+c)+24f^3\beta^3$$
$$+f^4\beta^2(52\lambda f-45c)-24f^5\beta+2cf^6\} = 0$$

which will be satisfied if either

$$\beta' = 0 \tag{4.4}$$

or

$$\beta^6(8\lambda f-3c)-3f^2\beta^4(4\lambda f+c)+24f^3\beta^3$$
$$+f^4\beta^2(52\lambda f-45c)-24f^5\beta+2cf^6 = 0. \tag{4.5}$$

Both the results (4.4) and (4.5) show that β is constant. We arrive at the similar conclusion after considering the equation (4.1*b*).

5. Field equations in Bonnor's theory

From (1.7), (2.1) and (2.2) we have for the nonvanishing components of U_{ij}

$$U_{11}=\frac{\alpha f^2}{\beta^2+f^2} \qquad U_{22}=\operatorname{cosec}^2\theta\ U_{33}=-\frac{\beta f^2}{\beta^2+f^2} \tag{5.1}$$
$$U_{44}=-\frac{\gamma f^2}{\beta^2+f^2} \qquad U_{\underset{\vee}{23}}=-\frac{f(f^2+2\beta^2)\sin\theta}{\beta^2+f^2}.$$

The field equations of Bonnor's theory therefore reduce to

$$R_{11}+p^2U_{11}=0 \qquad R_{22}+p^2U_{22}=\operatorname{cosec}^2\theta(R_{33}+p^2U_{33})=0$$
$$R_{44}+p^2U_{44}=0 \qquad R_{\underset{\vee}{23}}+p^2U_{\underset{\vee}{23}}=c\sin\theta. \tag{5.2}$$

Taking linear combinations of these equations as before we obtain

$$\frac{\gamma''}{\gamma}-\frac{\gamma'}{2\gamma}\left(\frac{\alpha'}{\alpha}+\frac{\gamma'}{\gamma}\right)+A\frac{\gamma'}{\gamma}+2\alpha\frac{p^2f^2}{\beta^2+f^2}=0 \tag{5.3a}$$

$$\beta''-\frac{\beta'}{2}\left(\frac{\alpha'}{\alpha}-\frac{\gamma'}{\gamma}\right)+\frac{2\alpha}{\beta^2+f^2}(3\beta f^2p^2-\beta^2+f^2+2\beta fc)=0 \tag{5.3b}$$

$$-f(\beta')^2+2\alpha\{f^2(c+p^2f)-\beta^2(c+2p^2f)-2\beta f\}=0 \tag{5.3c}$$

in addition to (3.2*a*). From (5.3*c*) and (3.3) we obtain

$$\alpha=\frac{f(\beta')^2}{2\{f^2(c+p^2f)-\beta^2(c+2p^2f)-2\beta f\}}$$
$$\gamma=\frac{2\beta^3\{f^2(c+p^2f)-\beta^2(c+2p^2f)-2\beta f\}}{c_1f(\beta^2+f^2)^2}.$$

Substituting α and γ in either of the equations (5.3*a*), (5.3*b*) we arrive at a result similar to the previous sections.

6. Conclusion

In general relativity we have the Nordström solution (Eddington 1924) representing the spherically symmetric vacuum electrostatic field due to an electron. To get its analogue in the unified field theories of Einstein (1953), Schrödinger (1947) and Bonnor (1954) we consider the static spherically symmetric total field of Papapetrou (1948). We find that the field equations have no nontrivial solution for the external field of an isolated charged pole. This indicates the absence of isolated charged monopoles in the theories considered.

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